

Gröbner-Shirshov bases for plactic algebras

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Abstract

A finite Gröbner-Shirshov basis is constructed for the plactic algebra of rank 3 over a field K . It is also shown that plactic algebras of rank exceeding 3 do not have finite Gröbner-Shirshov bases associated to the natural degree-lexicographic ordering on the corresponding free algebra. The latter is in contrast with the case of a strongly related class of algebras, called Chinese algebras, considered in [5].

Let P_n denote the plactic algebra of rank $n \geq 3$ over a field K . So $P_n = K[M_n]$ is the monoid algebra over K of the plactic monoid M_n of rank n , which is defined by the following finite presentation:

$$M_n = \langle x_1, \dots, x_n \mid R \rangle,$$

where R is the set of all relations (called Knuth relations) of the form

$$x_j x_i x_k = x_j x_k x_i, \quad x_i x_k x_j = x_k x_i x_j \text{ for } i < j < k$$

and

$$x_j x_i x_i = x_i x_j x_i, \quad x_j x_i x_j = x_j x_j x_i \text{ for } i < j.$$

The origin of the plactic monoid stems from Schensted's algorithm that was developed in order to determine the maximal length of a non-increasing subsequence and the maximal length of a decreasing subsequence in any finite sequence with elements in the set $\{1, 2, \dots, n\}$, see [11]. Combinatorics of $M = M_n$ was thoroughly studied in [11], see also [10]. In particular, it is known that elements of M admit a canonical normal form, expressed in terms of the associated Young tableaux. Later, deep applications of the plactic monoid to problems in representation theory, algebraic combinatorics, theory of quantum groups and relations to some other important areas of mathematics were discovered. We refer to [8], [10] and [12] for a presentation of these topics. Some ring theoretical properties of the plactic algebra were described in [4]. The purpose of this note is to study Gröbner-Shirshov bases of $K[M]$ related to the natural order on the associated free algebra.

Let X be the free monoid of rank n , with free generators also denoted by x_1, \dots, x_n . Then X is equipped with the degree-lexicographic order extending the following order on the generating set of X : $x_1 < x_2 < \dots < x_n$. For

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simplicity, the generators x_i will be also denoted by i , if unambiguous. This natural order is inherent in the nature of the plactic monoid, originally developed for the combinatorial problem mentioned above.

Our first result reads as follows.

Theorem 1 *If $n = 3$ then $K[M]$ has a finite Gröbner-Shirshov basis. Namely, the following elements, viewed as elements of the free algebra $K[X]$, form such a basis:*

- i) $332 - 323$
- ii) $322 - 232$
- iii) $331 - 313$
- iv) $311 - 131$
- v) $221 - 212$
- vi) $211 - 121$
- vii) $231 - 213$
- viii) $312 - 132$
- ix) $3212 - 2321$
- x) $32131 - 31321$
- xi) $32321 - 32132$.

Proof. Recall that by a reduction we mean a replacement of a subword of a given word w of the free monoid X that is the leading term of an element $f \in K[X]$ listed in i)-xi) by a subword that is equal to the remaining monomial w' of f . For example, $1321223 \rightarrow 1232123$ is a reduction using the polynomial $f = 3212 - 2321$ listed in ix) above. First, we list all ambiguities between two types of reductions that can occur in the process of bringing w to a form that cannot be reduced anymore (notice that such a form exists because the defining relations of M are homogeneous):

1. $(332)2 = 3(322)$
2. $(332)21 = 33(221)$
3. $(332)11 = 33(211)$
4. $(332)31 = 33(231)$
5. $(332)12 = 3(3212)$
6. $(332)131 = 3(32131)$
7. $(332)321 = 3(32321)$

8. $(322)1 = 3(221)$
9. $(322)21 = 32(221)$
10. $(322)11 = 32(211)$
11. $(322)31 = 32(231)$
12. $(331)1 = 3(311)$
13. $(331)2 = 3(312)$
14. $2(311) = (231)1$
15. $321(311) = (32131)1$
16. $(221)1 = 2(211)$
17. $31(221) = (312)21$
18. $321(221) = (3212)21$
19. $31(211) = (312)11$
20. $321(211) = (3212)11$
21. $323(211) = (32321)1$
22. $(231)2 = 2(312)$
23. $31(231) = (312)31$
24. $321(231) = (3212)31$
25. $321(312) = (32131)2$
26. $32(3212) = (32321)2$
27. $32(32131) = (32321)31$

Next, we show that all ambiguities can be resolved, using the reductions provided by the elements i)-xi) listed above (see the diamond lemma in [1], see also [6]).

1. $(332)2 \rightarrow 3232$ (using i))
 $3(322) \rightarrow 3232$ (using ii))
2. $(332)21 \rightarrow 32321$ (using i))
 $33(221) \rightarrow 33213 \rightarrow 32313$ (using v), i))
3. $(332)11 \rightarrow 32311 \rightarrow 32131 \rightarrow 31321$ (using i), vii), x))
 $33(211) \rightarrow 33121 \rightarrow 31321$ (using vi), iii))

4. $(332)31 \rightarrow 32331 \rightarrow 32313$ (using i, iii)
 $33(231) \rightarrow 33213 \rightarrow 32313$ (using vii, i))
5. $(332)12 \rightarrow 32312 \rightarrow 32132$ (using i, vii)
 $3(3212) \rightarrow 32321 \rightarrow 32132$ (using ix, xi))
6. $(332)131 \rightarrow 323131 \rightarrow 321331 \rightarrow 321313 \rightarrow 313213$ (using i, vii, iv, x)
 $3(32131) \rightarrow 331321 \rightarrow 313321 \rightarrow 313231 \rightarrow 313213$ (using xi, iii, i, vii))
7. $(332)321 \rightarrow 323321 \rightarrow 323231 \rightarrow 323213 \rightarrow 321323$ (using i, i, vii, xi)
 $3(32321) \rightarrow 332132 \rightarrow 323132 \rightarrow 321332 \rightarrow 321323$ (using xi, i, vii, i))
8. $(322)1 \rightarrow 2321$ (using ii)
 $3(221) \rightarrow 3212 \rightarrow 2321$ (using v, ix))
9. $(322)21 \rightarrow 23221 \rightarrow 22321$ (using ii, ii)
 $32(221) \rightarrow 32212 \rightarrow 23212 \rightarrow 22321$ (using v, ii, ix))
10. $(322)11 \rightarrow 23211$ (using i)
 $32(211) \rightarrow 32121 \rightarrow 23211$ (using vi, ix))
11. $(322)31 \rightarrow 23231 \rightarrow 23213$ (using ii, vii)
 $32(231) \rightarrow 32213 \rightarrow 23213$ (using vii, ii))
12. $(331)1 \rightarrow 3131$ (using iii)
 $3(311) \rightarrow 3131$ (using iv))
13. $(331)2 \rightarrow 3132$ (using iii)
 $3(312) \rightarrow 3132$ (using viii))
14. $2(311) \rightarrow 2131$ (using iv)
 $(231)1 \rightarrow 2131$ (using vii))
15. $321(311) \rightarrow 321131 \rightarrow 312131 \rightarrow 132131 \rightarrow 131321$ (using iv, vi, viii, x)
 $(32131)1 \rightarrow 313211 \rightarrow 313121 \rightarrow 311321 \rightarrow 131321$ (using x, vi, viii, iv))
16. $(221)1 \rightarrow 2121$ (using v)
 $2(211) \rightarrow 2121$ (using vi))
17. $31(221) \rightarrow 31212 \rightarrow 13212$ (using v, viii)
 $(312)21 \rightarrow 13221 \rightarrow 13212$ (using viii, v))

18. $321(221) \rightarrow 321212 \rightarrow 232112 \rightarrow 231212 \rightarrow 213212 \rightarrow 212321$ (using v),
 $(ix), vi), vii), ix))$
 $(3212)21 \rightarrow 232121 \rightarrow 223211 \rightarrow 223121 \rightarrow 221321 \rightarrow 212321$ (using
 $ix), ix), vi), vii), v))$
19. $31(211) \rightarrow 31121 \rightarrow 13121$ (using vi), iv))
 $(312)11 \rightarrow 13211 \rightarrow 13121$ (using viii), vi))
20. $321(211) \rightarrow 321121 \rightarrow 312121 \rightarrow 132121 \rightarrow 123211 \rightarrow 123121 \rightarrow 121321$
 $(using vi), vi), viii), ix), vi), viii))$
 $(3212)11 \rightarrow 232111 \rightarrow 231211 \rightarrow 213211 \rightarrow 213121 \rightarrow 211321 \rightarrow 121321$
 $(using ix), vi), vii), vi), viii), vi))$
21. $323(211) \rightarrow 323121 \rightarrow 321321$ (using vi), vii))
 $(32321)1 \rightarrow 321321$ (using xi))
22. $(231)2 \rightarrow 2132$ (using vii))
 $2(312) \rightarrow 2132$ (using viii))
23. $31(231) \rightarrow 31213 \rightarrow 13213$ (using vii), viii))
 $(312)31 \rightarrow 13231 \rightarrow 13213$ (using viii), vii))
24. $321(231) \rightarrow 321213 \rightarrow 232113 \rightarrow 231213 \rightarrow 213213$ (using vii), ix), vi),
vii))
 $(3212)31 \rightarrow 232131 \rightarrow 231321 \rightarrow 213321 \rightarrow 213231 \rightarrow 213213$ (using ix),
x), vii), i), vii))
25. $321(312) \rightarrow 321132 \rightarrow 312132 \rightarrow 132132$ (using viii), vi), viii))
 $(32131)2 \rightarrow 313212 \rightarrow 312321 \rightarrow 132321 \rightarrow 132132$ (using x), ix), viii),
xi))
26. $32(3212) \rightarrow 322321 \rightarrow 232321 \rightarrow 232132$ (using ix), ii), xi))
 $(32321)2 \rightarrow 321322 \rightarrow 321232 \rightarrow 232132$ (using xi), ii), ix))
27. $(32321)31 \rightarrow 3213231 \rightarrow 3213213$ (using x), vii))
 $32(32131) \rightarrow 3231321 \rightarrow 3213321 \rightarrow 3213231 \rightarrow 3213213$ (using xi), vii),
i), v))

The result follows. ■

Notice that the above result may be reformulated to say that M admits a complete rewriting system, see [2].

Let v, u be elements of the free monoid X with free generators $\{1, \dots, n\}$. If $v = v_m \cdots v_1, u = u_q \cdots u_1 \in X, v_j, u_i \in \{1, \dots, n\}$, then we write $v \prec u$ if $m \geq q$ and $v_i \leq u_i$ for $i = 1, \dots, q$. Recall from [10] that every $w \in M$ can be uniquely written in the form $w = w_1 w_2 \cdots w_k$ for some decreasing words

w_1, \dots, w_k such that $w_i \prec w_{i+1}$ for $i = 1, \dots, k-1$. This is called the standard tableaux form of w . For example, $(421)^2(31)(32)2^34$ is an element of $M = M_4$ written in this form.

The following is an easy consequence of Theorem 1.

Corollary 2 *The Gröbner-Shirshov basis found above leads to the following normal forms of elements of the plactic monoid of rank 3*

$$(1)^i(21)^j(2)^k(321)^l(32)^m(3)^q$$

or

$$(1)^i(21)^j(31)^k(321)^l(32)^m(3)^q$$

for non-negative integers i, j, k, l, m, q .

Proof. It is clear that words of the above two types cannot be reduced, using the reductions described in Theorem 1. Hence, it remains to show that every word can be reduced to one of the above forms; or, that every minimal word $w \in X$ (a word that cannot be reduced) is of one of the above forms. This can be easily seen because if $w \in \langle 1, 2 \rangle$ then w can be reduced to a word of the form $1^i(21)^j2^k$. Otherwise, write $w = u3v$, where $u, v \in X$. If $v \in 1M$ then $u \notin M2$, so u can be reduced to a word of the form $1^i(21)^j$. Otherwise, u can be reduced to $1^i(21)^j2^k$. Next, it is easy to see that $3v$ is of the form $(31)^p(321)^l(32)^m(3)^q$. Since 231 cannot be a subword of w , the assertion follows.

Another way of proving the claim is to show that the standard tableaux forms of the elements listed in the statement (they all must be different) exhaust all tableaux forms of elements of M . This is an easy consequence of the algorithm that allows to bring any word to a word in the tableau form, see [10] ■

We conclude with the following surprising observation.

Theorem 3 *If $n > 3$ then every Gröbner-Shirshov basis of $K[M]$ (associated to the degree-lexicographic ordering of M) is infinite.*

Proof. Consider the words $w_i = 323^i431 \in X$, for $i = 1, 2, \dots$. Since we have the following equalities in M :

$$323 = 332, 32431 = 34231 = 34213 = 32413 = 32143, 3213 = 3321$$

it follows that $323^i431 = w_i = 3213^i43$, holds also in M .

Let $a = 323^i43, b = 23^i431 \in X$. We claim that a is the minimal word in X among all words that represent 323^i43 as an element of M . We also claim that b is the minimal word in X among all words that represent 23^i431 as an element of M . Then it is clear that, in order to reduce the word 323^i431 , we have to include the reduction $323^i431 \rightarrow 3213^i43$ to the set of allowed reductions. Since $i \geq 1$ is arbitrary, the set of reductions must be infinite, and the assertion follows.

In order to verify the claims, notice first that the defining relations do not allow to rewrite 323^i43 in M in the form $2v$ for some $v \in X$, or in the form $u4$ for some $u \in X$. It follows easily that 323^i43 is a minimal word in its class in M .

Next, consider the word $b = 23^i431$. Suppose $b \in 1M$. The only defining relations that can be used to bring 1 to the first position in a presentation of b are $312 = 132$, $412 = 142$ or $413 = 143$. In the corresponding cases we would get $b = 1323^j43^{i-j}$, $b = 1423^{i+1}$, $b = 1433^j23^{i-j}$, for some $j \geq 0$, respectively. Since the maximal length of a decreasing subsequence of the given word is an invariant of the plactic class of this word, see [10], this implies that $b = 1433^j23^{i-j}$. However, the standard tableaux form of the latter element is easily seen to be $(431)23^i$, while the standard tableaux form of 23^i431 is $(421)3^{i+1}$, [10]. This contradiction shows that $b \notin 1M$. Suppose $b \in 21M$. Then $b = 213^j43^{i-j+1}$ in M , for some $j \geq 0$, again a contradiction because the latter has no decreasing subsequence of length three. So a minimal word that represents b in M should start with 23 and hence it must be of the form 23^i431 (because it should have a decreasing subsequence of length three). The result follows. ■

The above result is in contrast with the corresponding result for the strongly related class of monoids, also defined by homogeneous monoid presentations and with all defining relations of degree 3 - the so called Chinese monoids. The latter class was introduced and its combinatorial properties were studied in [3]. It was shown in [5] that the Gröbner-Shirshov basis of the Chinese algebra of any rank $n \geq 1$ is finite. Notice that the Chinese algebra of rank n has the same growth function (see [9]) as the plactic algebra of rank n and its elements also admit a canonical form expressed in terms of certain tableaux, [7]. Moreover, if $n < 3$, then the two algebras coincide.

References

- [1] G. M. Bergman, The diamond lemma for ring theory, *Adv. Math.* 29 (1978), 178–218.
- [2] R. V. Book, F. Otto, *String-Rewriting Systems*, (Springer-Verlag, New York, 1993).
- [3] J. Cassaigne, M. Espie, D. Krob, J.-C. Novelli and F. Hivert, The Chinese monoid, *Int. J. Algebra Comput.* 11 (2001), 301–334.
- [4] F. Cedó and J. Okniński, Plactic algebras, *J. Algebra* 274 (2004), 97–117.
- [5] Chen Yuqun and Qiu Jianjun, Gröbner-Shirshov basis for the Chinese monoid, *J. Algebra Appl.* 7 (2008), 623–628.
- [6] P. M. Cohn, *Algebra, Volume 3*, (2nd. ed., John Wiley & Sons, New York, 1991).
- [7] G. Duchamp and D. Krob, Plactic-growth-like monoids, in: *Words, languages and combinatorics II*, pp. 124–142 (World Scientific, Singapore, 1994).

- [8] W. Fulton, *Young Tableaux* (Cambridge University Press, New York, 1997).
- [9] G. R. Krause and T. H. Lenagan, *Growth of Algebras and Gelfand-Kirillov Dimension* (Revised ed., Graduate Studies in Mathematics, 22, American Mathematical Society, Providence, RI, 2000).
- [10] A. Lascoux, B. Leclerc and J. Y. Thibon, The plactic monoid, *in: Combinatorics on Words, Chapter 5* (Cambridge Univ. Press, 2002).
- [11] A. Lascoux and M. P. Schützenberger, Le monoïde plaxique, *in: Noncommutative structures in algebra and geometric combinatorics (Naples, 1978)*, pp. 129–156 (Quad. “Ricerca Sci.”, 109, CNR, Rome, 1981).
- [12] B. Leclerc and J.-Y. Thibon, The Robinson-Schensted correspondence, crystal bases, and the quantum straightening at $q = 0$, *J. Comb.* 3 (1996), 245–268.

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